Souvenir J: UMP Tests

Here are some familiar Uniformly Most Powerful (UMP) unbiased tests that fall out of the Λ criterion (letting μ be the mean):

(1) One-sided Normal test. Each X_i is NIID, $N(\mu, \sigma^2)$, with σ known: H_0: $\mu \leq \mu_0$ against H_1: $\mu > \mu_0$.

$$d(\boldsymbol{X}) = \sqrt{n}(\overline{X} - \mu_0)/\sigma, \ RR(\alpha) = \{\boldsymbol{x}: d(\boldsymbol{x}) \geq c_\alpha\}.$$

Evaluating the Type I error probability requires the distribution of $d(\boldsymbol{X})$ under H_0: $d(\boldsymbol{X}) \sim N(0,1)$.

Evaluating the Type II error probability (and power) requires the distribution of $d(\boldsymbol{X})$ under $H_1[\mu = \mu_1]$:

$$d(\boldsymbol{X}) \sim N(\delta_1,1), \text{ where } \delta_1 = \sqrt{n}(\mu_1 - \mu_0)/\sigma.$$

(2) One-sided Student's t test. Each X_i is NIID, $N(\mu, \sigma^2)$, σ unknown: H_0: $\mu \leq \mu_0$ against H_1: $\mu > \mu_0$:

$$d(\boldsymbol{X}) = \sqrt{n}(\overline{X} - \mu_0)/s, \quad RR(\alpha) = \{\boldsymbol{x}: d(\boldsymbol{x}) \geq c_\alpha\},$$

$$s^2 = \left[\frac{1}{(n-1)}\right]\sum(X_i - \overline{X})^2.$$

Two-sided Normal test of the mean H_0: $\mu = \mu_0$ against H_1: $\mu \neq \mu_0$:

$$d(\boldsymbol{X}) = \sqrt{n}(\overline{X} - \mu_0)/s, \quad RR(\alpha) = \{\boldsymbol{x}: |d(\boldsymbol{x})| \geq c_\alpha\}.$$

Evaluating the Type I error probability requires the distribution of $d(\boldsymbol{X})$ under H_0: $d(\boldsymbol{X}) \sim St(n-1)$, the Student's t distribution with $(n-1)$ degrees of freedom (df).

Evaluating the Type II error probability (and power) requires the distribution of $d(\boldsymbol{X})$ under $H_1[\mu = \mu_1]$: $d(\boldsymbol{X}) \sim St(\delta_1, n-1)$, where $\delta_1=\sqrt{n}(\mu_1 - \mu_0)/\sigma$ is the non-centrality parameter.

This is the UMP, unbiased test.

(3) The difference between two means (where it is assumed the variances are equal):

$H_0: \gamma := \mu_1 - \mu_2 = \gamma_0$ against $H_1: \gamma_1 \neq \gamma_0$.

A Uniformly Most Powerful Unbiased (UMPU) test is defined by

$$\tau(\mathbf{Z}) = \frac{\sqrt{n}\left[(\overline{X}_n - \overline{Y}_n) - \gamma_0\right]}{s\sqrt{2}}, \text{RR} = \left\{\mathbf{z}: |\tau(\mathbf{z})| \geq c_\alpha\right\}.$$

Under H_0: $\tau(\mathbf{Z}) = \dfrac{\sqrt{n}\left[(\overline{X}_n - \overline{Y}_n) - \gamma_0\right]}{s\sqrt{2}} \sim \text{St}(2n-2)$,

under $H_1[\gamma = \gamma_1]$: $\tau(\mathbf{Z}) \sim \text{St}(\delta_1; 2n-2)$, $\delta_1 = \dfrac{\sqrt{n}\,(\gamma_1 - \gamma_0)}{\sigma\sqrt{2}}$, for $\gamma_1 \neq \gamma_0$.

Many excellent sources of types of tests exist, so I'll stop with these.